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# CONVEX EMBEDDINGS AND BISECTIONS OF 3-CONNECTED GRAPHS[1]

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Given two disjoint subsets $T_1$ and $T_2$ of nodes in an undirected 3-connected graph $G=(V,E)$ with node set $V$ and arc set $E$, where $|T_1|$ and $|T_2|$ are even numbers, we show that $V$ can be partitioned into two sets $V_1$ and $V_2$ such that the graphs induced by $V_1$ and $V_2$ are both connected and $|V_1 \cap T_j|=|V_2 \cap T_j|=|T_j|/2$ holds for each $j=1,2$. Such a partition can be found in $O(|V|^2 \log |V|)$ time. Our proof relies on geometric arguments. We define a new type of 'convex embedding' of $k$-connected graphs into real space $\mathbf{R}^{k-1}$ and prove that for $k=3$ such an embedding always exists.

## 1. Introduction

We define the following graph-partitioning problem: Given an undirected graph $G=(V,E)$ and $k$ subsets $T_1,\dots,T_k$ of $V$, not necessarily disjoint, find a partition $V$ into $l$ subsets $V_1,\dots,V_l$ such that $G[V_i]$ $(1 \le i \le l)$ are all connected and $a_{ij} \le |V_i \cap T_j| \le b_{ij}$ holds for each pair $i,j$, where $a_{ij}$ and $b_{ij}$ are prespecified lower and upper bounds. In this problem, we interpret each $T_i$ as a set of nodes which possess 'resource' $i$. In particular, one may ask for a partition where the nodes in each $T_j$ $(1 \le j \le k)$ are distributed among the subsets $V_1,\dots,V_l$ as equally as possible.

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In this paper, we consider the case of $l = 2$ and ask to distribute the resources equally: Given an undirected graph $G = (V, E)$ and $k$ subsets $T_1, T_2, \ldots, T_k$ of $V$, where $|T_j|$ is even for all $j$, find a bipartition $\{V_1, V_2\}$ of $V$ such that the graph $G[V_i]$ induced by each $V_i$ is connected and $|V_1 \cap T_j| = |V_2 \cap T_j|$ $(= |T_j|/2)$ holds for $j = 1, \ldots, k$. Let us call this problem *the k-bisection problem*. We prove that for every 3-connected graph $G$ and for every choice of (disjoint) resource sets $T_1$ and $T_2$, the 2-bisection problem has a solution.

To verify this, we reduce it to a geometrical problem. Our method is outlined as follows. We first prove that every 3-connected graph $G$ can be embedded in the plane in such a way that the convex hull of its nodes, which is a convex polygon, corresponds to a cycle $C$ of $G$, and every node $v$ in $V - V(C)$ is in the convex hull of its neighbors (precise definition is given in Section 4). This will guarantee that, for any given straight line $L$ in the plane, each of the two subgraphs of $G$ separated by $L$ remains connected. Given such an embedding, we apply the 'ham-sandwich cut' algorithm, which is well known in computational geometry, to find a straight line $L^*$ that bisects the two subsets $T_1$ and $T_2$ simultaneously. Since the above embedding ensures that the two subgraphs separated by the ham-sandwich cut $L^*$ are connected, this bipartition of the nodes becomes a solution to the 2-bisection problem.

We give an algorithm which finds such a bisection in $O(|V|^2 \log |V|)$ time.

### 1.1. Related results

If $l = 2$ and there is just one set $T$ of resources, the problem is NP-hard for general graphs, since it is NP-hard to test whether a given graph $G = (V, E)$ and an integer $n_1 < |V|$ have a partition of $V$ into two subsets $V_1$ and $V_2$ such that the graph induced by $V_i$ is connected for $i = 1, 2$, and $|V_1| = n_1$ holds [3,4]. When $G$ is 2-connected, it is known that such a partition $\{V_1, V_2\}$ always exists and it can be found in linear time [14,16]. More generally, the following result was shown independently by Győri [8] and Lovász [11].

**Theorem 1.1.** [8,11] *Let $G = (V, E)$ be an $\ell$-connected graph, $w_1, w_2, \ldots, w_\ell \in V$ be different nodes and $n_1, n_2, \ldots, n_\ell$ be positive integers such that $n_1 + n_2 + \ldots + n_\ell = |V|$. Then there exists a partition $\{V_1, V_2, \ldots, V_\ell\}$ of $V$ such that $G[V_i]$ is connected, $|V_i| = n_i$ and $w_i \in V_i$ for $i = 1, \ldots, \ell$.* ∎

A slight extension of this theorem is obtained by Wada *et al.* [17].

**Theorem 1.2.** [17] *Let $G = (V, E)$ be an $\ell$-connected graph, $T$ be a subset of $V$, $w_1, \ldots, w_\ell \in T$ be different nodes and $n_1, \ldots, n_\ell$ be positive integers* *such that* $n_1+\ldots+n_\ell=|T|$. *Then there exists a partition* $\{V_1,\ldots,V_\ell\}$ *of* $V$ *such that* $G[V_i]$ *is connected,* $|V_i\cap T|=n_i$ *and* $w_i\in V_i$ *for* $i=1,\ldots,\ell$. ∎

## 2. Preliminaries

Let $G=(V,E)$ stand for an undirected graph with a set $V$ of *nodes* and a set $E$ of *arcs*, where we denote $|V|$ by $n$, $|E|$ by $m$. For a subgraph $H$ of $G$, the sets of nodes and arcs in $H$ are denoted by $V(H)$ and $E(H)$, respectively. Let $X$ be a subset of $V$. The subgraph of $G$ induced by $X$ is denoted by $G[X]$. A node $v\in V-X$ is called a *neighbor* of $X$ if it is adjacent to some node $u\in X$, and the set of all neighbors of $X$ is denoted by $N_G(X)$. Let $e=(u,v)$ be an arc with end nodes $u$ and $v$. We denote by $G/e$ the graph obtained from $G$ by contracting $u$ and $v$ into a single node (deleting any resulted self-loop), and by $G-e$ the graph obtained from $G$ by removing $e$. *Subdividing* an arc $e=(u,v)$ means that we replace $e$ by a path $P$ from $u$ to $v$ where the inner nodes of $P$ are new nodes of the graph. If we obtain a graph $G'$ by subdividing some arcs in $G$, then the resulted graph is called a *subdivision* of $G$. A graph $G$ is *$k$-connected* if and only if $|V|\geq k+1$ and the graph $G-X$ obtained from $G$ by removing any set $X$ of $(k-1)$ nodes remains connected. A singleton set $\{x\}$ may be simply written as $x$.

### 2.1. The Ham-Sandwich Theorem

Consider the $d$-dimensional space $\mathbf{R}^d$. For a non-zero $a\in\mathbf{R}^d$ and a real $b\in\mathbf{R}^1$, $H(a,b)=\{x\in\mathbf{R}^d\mid\langle a\cdot x\rangle=b\}$ is called a *hyperplane*, where $\langle a\cdot x\rangle$ denotes the inner product of $a,x\in\mathbf{R}^d$. Moreover, $H^+(a,b)=\{x\in\mathbf{R}^d\mid\langle a\cdot x\rangle\geq b\}$ (resp., $H^-(a,b)=\{x\in\mathbf{R}^d\mid\langle a\cdot x\rangle\leq b\}$) is called a *positive closed half space* (resp., *negative closed half space*) with respect to $H=H(a,b)$.

Let $P_1,\ldots,P_d$ be $d$ sets of points in $\mathbf{R}^d$. We say that a hyperplane $H=H(a,b)$ in $\mathbf{R}^d$ *bisects* $P_i$ if $|H^+(a,b)\cap P_i|\geq\left\lceil\frac{|P_i|}{2}\right\rceil$ and $|H^-(a,b)\cap P_i|\geq\left\lceil\frac{|P_i|}{2}\right\rceil$. Thus, if $|P_i|$ is odd, then any bisector $H$ of $P_i$ contains at least one point of $P_i$. If $H$ bisects $P_i$ for all $i=1,\ldots,d$, then $H$ is called a *ham-sandwich cut* with respect to the sets $P_1,\ldots,P_d$. The following theorem is well-known.

**Theorem 2.1.** [5] *Given* $d$ *sets* $P_1,\ldots,P_d$ *of points in the* $d$*-dimensional space* $\mathbf{R}^d$, *there exists a hyperplane which is a ham-sandwich cut with respect to* $P_1,\ldots,P_d$. ∎

Edelsbrunner and Waupotitsch [6] proposed an

$$O(|P_1 \cup P_2| \log(\min\{|P_1|, |P_2|\} + 1))$$

time algorithm that computes a ham-sandwich cut of $P_1$ and $P_2$ in $\mathbf{R}^2$. Afterwards Chi-Yuan, Matoušek and Steiger [2] proposed algorithms for finding a ham-sandwich cut in $O(|P_1 \cup P_2|)$ time for $d=2$ and $O(|P_1 \cup P_2|^{3/2})$ time for $d=3$, respectively.

## 3. Bisecting $k$ Subsets in $(k+1)$-connected Graphs

Let $G=(V,E)$ be a graph and $T_1, T_2, \ldots, T_k$ be subsets of $V$, where $|T_j|$ is even for $j=1,\ldots,k$. A bipartition $\{V_1, V_2\}$ of $V$ is a *$k$-bisection* (or *weak $k$-bisection*) if $G[V_i]$ is connected for $i=1,2$, and $|V_1 \cap T_j| = |V_2 \cap T_j|$ $(=|T_j|/2)$ holds for $j=1,2,\ldots,k$ ($|V_i \cap T_j| \leq (|T_j|+k)/2$ holds for $i=1,2$ and $j=1,2,\ldots,k$, respectively). Our goal is to find best possible sufficient conditions for the existence of (weak) $k$-bisections, in terms of the connectivity of $G$. In the following examples we show families of highly connected graphs with specified sets $T_i$ which possess no $k$-bisections.

**Example 1.** Let $G_k$ ($k \in \{1,2,3\}$) be the $k$-connected graph obtained by taking three copies of $K_k$ and one copy of $\bar{K}_k$, and connecting each copy of $K_k$ to $\bar{K}_k$ by $k$ independent edges. Let each of the $k$ pairwise disjoint specified sets $T_j$ ($j=1,\ldots,k$) consist of a node $v_j$ of $\bar{K}_k$ and the three neighbours of $v_j$. It is easy to check that $G_k$ has no $k$-bisection for $1 \leq k \leq 3$.

**Example 2.** Let $k \geq 3$ and consider the following graph. Let $K_{2k-1,k} = (W \cup Z, E)$ be a complete bipartite graph with node sets $W = \{w_1, w_2, \ldots, w_{2k-1}\}$ and $Z = \{z_1, z_2, \ldots, z_k\}$, and let $T_i = W \cup \{z_i\}$ for $i=1,\ldots,k$. Note that $|T_i| = 2k$ holds for all $i=1,\ldots,k$. Suppose that $\{V_1, V_2\}$ is a $k$-bisection to $K_{2k-1,k}$ and $\{T_1, \ldots, T_k\}$. The set $T_1$ is bisected by $\{V_1, V_2\}$; $V_1 \cap T_1 = \{w_1, \ldots, w_k\}$ and $V_2 \cap T_1 = \{w_{k+1}, \ldots, w_{2k-1}, z_1\}$ can be assumed without loss of generality. Since $V_1 \cap T_i = \{w_1, \ldots, w_k\}$ also holds for each $i=2,\ldots,k$, $V_1$ cannot contain any node in $Z$ and hence $V_1$ must be $\{w_1, \ldots, w_k\}$. However the induced subgraph $G[V_1]$ is not connected. Thus these graphs admit no $k$-bisection.

Also notice that if $k \geq 3$ and $T_1 = \{v_1, v_2\}$, $T_2 = \{v_2, v_3\}$ and $T_3 = \{v_3, v_1\}$ for some three nodes $v_1, v_2, v_3 \in V$, then there exists no $k$-bisection. These examples show that $k$-connectivity is not sufficient to guarantee a $k$-bisection for arbitrarily specified sets $T_i$, even if the specified sets are pairwise disjoint. Furthermore, for $k \geq 3$, either we have to assume that the specified sets are pairwise disjoint or we have to look for weak $k$-bisections only. We propose the following conjecture.

**Conjecture 3.1.** Let $G = (V,E)$ be a $(k+1)$-connected graph and $T_1,T_2,\ldots,T_k$ be specified subsets of $V$, where $|T_j|$ is even for $j = 1,\ldots,k$. Then (a) $G$ has a weak $k$-bisection, and (b) if the $k$ specified sets are pairwise disjoint then $G$ has a $k$-bisection. ∎

Note that Theorem 1.2 implies that Conjecture 3.1 is true for $k=1$ (by setting $\ell = 2$, $T = T_1$ and $n_1 = n_2 = |T|/2$ in Theorem 1.2). In fact, such a partition in Theorem 1.2 with $\ell = 2$ can be computed in linear time by using the so-called '$st$-numbering' of nodes [16]. Thus, we have the following result.

**Corollary 3.2.** *Let $G=(V,E)$ be a 2-connected graph and $T$ be a subset of $V$ with even $|T|$. Then $G$ has a* 1*-bisection $\{V_1,V_2\}$, and such $\{V_1,V_2\}$ can be computed in $O(m)$ time.* ∎

The main result of this paper is an algorithmic proof for Conjecture 3.1 in the case of $k=2$. Our algorithm finds the required 2-bisections in a 3-connected graph in $O(n^2 \log n)$ time.

## 4. Strictly Convex Embeddings

In this section we introduce a new way of embedding a graph $G$ in $\mathbf{R}^d$. The existence of such an embedding (along with the ham-sandwich cut theorem) will play a crucial role in the proof of Conjecture 3.1 for $k=2$ in Section 5.

For a set $P = \{x_1,\ldots,x_k\}$ of points in $\mathbf{R}^d$, a point $x' = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$ with $\sum_{i=1,\ldots,k} \alpha_i = 1$ and $\alpha_i \geq 0$, $i = 1,\ldots,k$ is called a *convex combination* of $P$, and the set of all convex combinations of $P$ is denoted by $conv(P)$. If $P = \{x_1,x_2\}$, then $conv(P)$ is called a *segment* (connecting $x_1$ and $x_2$), denoted by $[x_1,x_2]$. A subset $S \subseteq \mathbf{R}^d$ is called a *convex set* if $[x,x'] \subseteq S$ for any two points $x,x' \in S$. For a convex set $S \subseteq \mathbf{R}^d$, a point $x \in S$ is called a *vertex* if there is no pair of points $x',x'' \in S - x$ such that $x \in [x',x'']$. For two vertices $x_1,x_2 \in S$, the segment $[x_1,x_2]$ is called an *edge* of $S$ if $\alpha x' + (1-\alpha)x'' = x \in [x_1,x_2]$ for some $0 \leq \alpha \leq 1$ implies $x',x'' \in [x_1,x_2]$. The intersection $S$ of a finite number of closed half spaces is called a *convex polyhedron*, and is called a *convex polytope* if $S$ is non-empty and bounded.

Given a convex polytope $S$ in $\mathbf{R}^d$, the *vertex-edge graph* $G_S = (V_S,E_S)$ is defined to be an undirected graph with node set $V_S$ corresponding to the vertices of $S$ and arc set $E_S$ corresponding to those pairs of vertices $x,x'$ for which $[x,x']$ is an edge of $S$. For a convex polyhedron $S$, a hyperplane $H(a,b)$ is called a *supporting hyperplane* of $S$ if $H(a,b) \cap S \neq \emptyset$ and either

$S \subseteq H^+(a,b)$ or $S \subseteq H^-(a,b)$. We say that a point $p \in S$ is *strictly inside* $S$ if there is no supporting hyperplane $H$ of $S$ containing $p$. If $S$ has a point strictly inside $S$ in $\mathbf{R}^d$, then $S$ is called *full-dimensional* in $\mathbf{R}^d$. The set of points strictly inside $conv(P)$ is denoted by $int(conv(P))$.

Given a graph $G = (V,E)$, an *embedding* of $G$ in $\mathbf{R}^d$ is a mapping $f : V \to \mathbf{R}^d$, where each node $v$ is represented by a point $f(v) \in \mathbf{R}^d$, and each arc $e = (u,v)$ by a segment $[f(u), f(v)]$ (which may be written by $f(e)$). For two arcs $e, e' \in E$, segments $f(e)$ and $f(e')$ may cross each other. For $\{v_1, v_2, \ldots, v_p\} = Y \subseteq V$, we denote by $f(Y)$ the set $\{f(v_1), \ldots, f(v_p)\}$ of points. For a set $Y$ of nodes, we denote $conv(f(Y))$ by $conv_f(Y)$.

We define a new kind of 'convex embedding' of a graph $G$ in the $d$-dimensional space:

**Definition 4.1.** Let $G = (V,E)$ be a graph without isolated nodes and let $G' = (V',E')$ be a subgraph of $G$. A *strictly convex embedding* (or *SC-embedding*, for short) of $G$ with *boundary* $G'$ is an embedding $f$ of $G$ into $\mathbf{R}^d$ in such a way that

(i) the vertex-edge graph of the full-dimensional convex polytope $conv_f(V')$ is isomorphic to $G'$ (such that $f$ itself defines an isomorphism),
(ii) $f(v) \in int(conv_f(N_G(v)))$ holds for all nodes $v \in V - V'$,
(iii) the points of $\{f(u) \mid u \in V\}$ are in general position.

It can be seen that the above definition implies that the vertices of $conv_f(V)$ are precisely the points in $f(V(G'))$.

A similar concept of 'convex embeddings' of graphs, requiring only (ii) above, was introduced by Linial *et al.* [10] and led to a new characterization of $k$-connected graphs. If $d = 1$, the embedding of [10] and the strictly convex embedding defined here are both equivalent to the so-called $s$-$t$-numberings. For higher dimensions the two concepts are different and the embedding of [10] does not seem sufficient for our purposes. An SC-embedding in the plane is illustrated by Figure 1.
SC-embeddings into $\mathbf{R}^d$ have the following important property. (In our proofs later we will use this property only in the special case $d = 2$.)

**Lemma 4.2.** *Let $G = (V,E)$ be a graph without isolated nodes and let $f$ be an SC-embedding of $G$ into $\mathbf{R}^d$. Let $f(V_1) \subseteq H^+(a,b)$ and $f(V) \cap (H^+(a,b) - H(a,b)) \subseteq f(V_1)$ hold for some hyperplane $H = H(a,b)$ and for some $\emptyset \neq V_1 \subseteq V$. Then $G[V_1]$ is connected.*

**Proof.** Let $G' = (V',E')$ be the boundary of $f$. The vertices of the convex polytope $S = conv_f(V)$ are the points in $f(V')$. By definition, $G'$ is isomorphic to the vertex-edge graph of $S$.

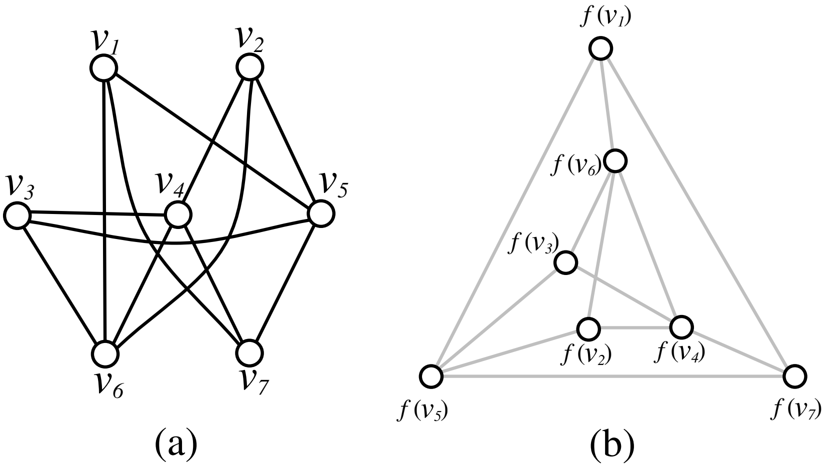


**Fig. 1.** *A* 3*-connected graph* $G_1$ *and an SC-embedding* $f$ *of* $G_1$ *with boundary* $C=(\{v_1,v_5,v_7\},\{(v_1,v_5),(v_5,v_7),(v_1,v_7)\})$

We first verify that $G[V_1\cap V']$ is connected. This fact follows easily from well-known properties of the simplex algorithm, but for completeness we provide another proof. The proof is by induction on $d$. For $d=1$ the statement is trivial. Suppose $d\geq 2$ and let $f(x)$ be a vertex of $S\cap H^+(a,b)$ in $f(V_1\cap V')$ which is furthest from $H$ (or equivalently, for which $\langle -a\cdot f(x)\rangle$ is maximum). Let $H_x$ be the hyperplane which is parallel to $H$ and contains $f(x)$. For a contradiction, suppose that some $y\in V_1\cap V'$ is not reachable from $x$ in $G[V_1\cap V']$. Choose $y$ in such a way that $f(y)$ is as far from $H$ as possible and let $H_y$ be a hyperplane which is parallel to $H$ and contains $f(y)$. The choice of $y$ implies that for every neighbor $z\in N_{G'}(y)$ we have $f(z)\in H_y^-$. If $H_x=H_y$, then $f(x)$ and $f(y)$ are both vertices of the $(d-1)$-dimensional polytope $S\cap H_x$ whose vertex-edge graph $S'$ is isomorphic to the corresponding subgraph $\hat{G}$ of $G[V_1\cap V']$ by the choice of $x$ and $y$. By induction $\hat{G}$ is connected, a contradiction.

Now assume that $H_y$ is strictly closer to $H$ than $H_x$. Using the fact that the cone $C$ generated by the edges incident to $f(y)$ contains $S$ (see [18, Lemma 3.6]), we obtain $S\subseteq C\subseteq H_y^-$. On the other hand, $f(x)\in H_y^+-H_y$ holds, a contradiction. This proves that $G[V_1\cap V']$ is connected.

Now we prove that $G[V_1]$ is connected. From the connectivity of $G[V_1\cap V']$ it is clear that if $G[V_1]$ is not connected, then $G[V_1]$ has a component $G[X]$ with $X\subseteq V_1-V'$. Consider a supporting hyperplane $H_X$ of $conv_f(X)$ such that $f(X)\subseteq H_X^-$ and $H_X$ is parallel to $H$. At least one point $f(x)$ in $f(X)$

is on $H_X$, but no point exists on one side of $H_X$. Since the node $x$ has no neighbors among the nodes mapped into that side of $H_X$, we have $f(x) \notin int(conv_f(N_G(x)))$. This contradicts the fact that $f$ is an SC-embedding of $G$. ∎

In what follows we assume that $d = 2$, that is, we investigate SC-embeddings in the plane $\mathbf{R}^2$ only, unless stated otherwise.

## 5. SC-embeddings in the Plane

In this section we prove that every 3-connected graph $G$ admits an SC-embedding $f$ with an arbitrarily specified cycle $C$ as its boundary, and show how to find such an SC-embedding efficiently. To do this, we use the following characterization of 3-connected graphs, due to Tutte.

**Lemma 5.1.** [15] *Let $G=(V,E)$ be a 3-connected graph. For any arc $e$, either $G/e$ is 3-connected or $G-e$ is a subdivision of a 3-connected graph.* ∎

For two points $x$ and $y$ in $\mathbf{R}^2$ let $L(x,y)$ denote the half line obtained by extending the segment $[x,y]$ in the direction from $x$ to $y$, and let $\hat{L}(x,y)$ denote the half line obtained from $L(x,y)$ by removing the points in $[x,y]-y$. That is, $L(x,y)=\{x+\alpha(y-x)\,|\,\alpha\geq 0\}$ and $\hat{L}(x,y)=\{x+\alpha(y-x)\,|\,\alpha\geq 1\}$.

Let $f$ be an SC-embedding of a graph $G=(V,E)$ in the plane with boundary $C$. We define a set $cone_f(v,u)\subseteq \mathbf{R}^2$ for each pair $(u,v)$ of adjacent nodes as follows. If $u\in V(C)$ or $f(u)\in int(conv_f(N_G(u)-\{v\}))$, then let $cone_f(v,u)=\mathbf{R}^2$. Otherwise $u\in V-V(C)$ and $f(u)$ is a vertex of $conv_f(N_G(u)\cup\{u\}-\{v\})$, and there are two arcs $e_1=(u,w_1)$ and $e_2=(u,w_2)$ with $w_1,w_2\in N_G(u)-\{v\}$ such that each of $f(e_1)=[f(u),f(w_1)]$ and $f(e_2)=[f(u),f(w_2)]$ is an edge of $conv_f(N_G(u)\cup\{u\}-\{v\})$. (See Figure 2.) In this case let $cone_f(v,u)$ be the interior of the cone bounded by the two half lines $\hat{L}(f(w_1),f(u))$ and $\hat{L}(f(w_2),f(u))$.

Now fix a node $v\in V-V(C)$ and let $f'$ be another embedding of $G$ such that $f'(u)=f(u)$ for all $u\in V-\{v\}$. We wish to find all those possible positions of $f'(v)$ for which

(1) $\quad f'(u)\in int(conv_{f'}(N_G(u)))$ for all nodes $u\in V-V(C)-\{v\}$.

Clearly, the set of good positions of $f'(v)$ depends only on the neighbors of $v$. Consider the following set.

(2) $$B_f(v)=\bigcap_{u\in N_G(v)} cone_f(v,u)$$

Observe that $B_f(v)$ is a non-empty open set, since $cone_f(v,u)$ is an open set containing $f(v)$, for each $u \in N_G(v)$. It is easy to see that $f'$ satisfies (1) if and only if $f'(v) \in B_f(v)$. Note that $cone_f(v,u)$ can be obtained in $O(|N_G(u)|)$ time since $int(conv_f(N_G(u) - \{v\}))$ can be computed in $O(|N_G(u)|)$ time, provided that the cyclic order of the points $\{f(w) \mid w \in N(u)\}$ around $u$ is known. Summarizing the above argument gives the next lemma.

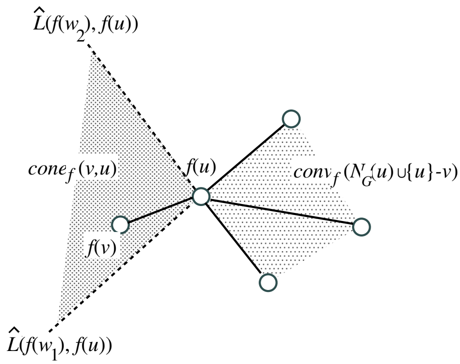


**Fig. 2.** *An example of* $cone_f(v,u)$

**Lemma 5.2.** *Given an SC-embedding $f$ of a graph $G=(V,E)$ with boundary $C$ and a node $v \in V - V(C)$, the set $B_f(v)$ is a non-empty open set. Furthermore, any embedding $f'$ with $f'(u)=f(u)$, $u \neq v$, and $f'(v) \in B_f(v)$ satisfies (1). Moreover $B_f(v)$ can be obtained in $O(\sum_{u \in N_G(v)-V(C)} |N_G(u)|)$ time.* ∎

We are ready to prove the main result of this section.

**Theorem 5.3.** *For every 3-connected graph $G=(V,E)$ and every cycle $C$ of $G$, there exists an SC-embedding with boundary $C$. Such an SC-embedding can be found in $O(n^2 \log n)$ time.*

**Proof.** We start with the outline of our constructive proof and some algorithmic remarks. First we compute a 3-connected spanning subgraph $G' = (V, E')$ of $G$ with $E(C) \subset E'$ and $|E'| = O(n)$. Such a 'sparse' spanning subgraph exists and can be found in linear time [12]. Clearly, an SC-embedding of $G'$ is also an SC-embedding of $G$.

We call an arc $e = (u,v)$ *internal* if $\{u,v\} \cap V(C) = \emptyset$. To find an SC-embedding of $G'$ first we apply the following procedure. We choose an arbitrary internal arc $e$ in $G'$. If $G' - e$ is not a subdivision of a 3-connected graph, then we contract the arc $e$. By Lemma 5.1 the resulted graph, denoted by $G/e$, is 3-connected. Otherwise, if $G'-e$ is a subdivision of a 3-connected graph, we delete the arc $e$ from $G'$, and, if there exist (one or two) nodes $v$ whose degree is two in $G'-e$, we remove every such $v$ as well after replacing the two incident arcs $(u_1,v)$ and $(v,u_2)$ with a single arc $(u_1,u_2)$. In this case let $G'\triangle e$ denote the resulted graph, which is clearly 3-connected. We repeat this procedure until there exists no internal arc in the current graph $H$. An SC-embedding of $H$ is easy to find: we embed the nodes in $V(C)$ in such a way that the segments corresponding to the arcs of the cycle $C$ form a convex polygon consisting of points in general position in the plane. Since there are no internal arcs, points of the remaining nodes of $H$ can be found independently (as far as new points together with the points in $C$ are in general position). This can be done in $O(n^2 \log n)$ time by choosing each new point in $O(n \log n)$ time so as to avoid creating a set of three points in a straight line.

Note that obtaining $H$ requires $O(n)$ 3-connectivity tests on graphs with $O(n)$ arcs each (due to the sparsification). Since a 3-connectivity test can be done in linear time [9], the total time to execute the above procedure is $O(n^2 \log n)$.

Finally, starting from $H$, we insert the contracted or deleted arcs and nodes one by one, in the reverse order of the above procedure. In every iteration we modify the current embedding in such a way that we maintain an SC-embedding of the graph after every insertion. This way we obtain an SC-embedding of $G'$. It remains to show how to construct an SC-embedding $f'$ of a graph $H'$, assuming that an SC-embedding $f$ for a 3-connected graph $H=H'\triangle e$ or $H=H'/e$ is available, where $e=(v_1,v_2)$ is an internal arc of $H'$.

**Case (i)**: $H=H'/e$.

Let $v^*$ be the node in $H'/e$ created by contracting the internal arc $e=(v_1,v_2)$. To regain $H'$, we insert $e$ to $H$ by splitting $v^*$ into $v_1$ and $v_2$ and connecting the appropriate pairs of nodes. We will define an SC-embedding $f$ of $H'$ with $f'(u)=f(u)$, $u\in V(H')-\{v_1,v_2\}$, and with $f'(v_1)=f(v^*)$. To determine $f'(v_2)$ we proceed as follows. Let $N_1=N_{H'}(v_1)-\{v_2\}$ and let $N_2=N_{H'}(v_2)-\{v_1\}$. If $f(v^*)\in \mathit{int}(\mathit{conv}_f(N_1))$ (and hence $f'(v_1)=f(v^*)\in \mathit{int}(\mathit{conv}_{f'}(N_1)))$, then we define $D=\mathbf{R}^2$. Otherwise, if $f(v^*)\notin \mathit{int}(\mathit{conv}_f(N_1))$, then $f(v^*)$ is a vertex of $\mathit{conv}_f(N_1\cup\{v^*\})$, and there are two arcs $e_1=(v^*,w_1)$ and $e_2=(v^*,w_2)$ with $w_1,w_2\in N_1$ such that each of $f(e_1)$ and $f(e_2)$ is an edge of $\mathit{conv}_f(N_1\cup\{v^*\})$. In this case let $D$ be the interior of the cone bounded by the two half lines $\hat{L}(f(w_1),f(v^*))$ and $\hat{L}(f(w_2),f(v^*))$.

Note that $D$ is a non-empty open set. By choosing $f'(v_2) \in D$, we have $f'(v_1)$ $(= f(v^*)) \in int(conv_{f'}(N_{H'}(v_1)))$.

To satisfy $f'(v_2) \in int(conv_{f'}(N_{H'}(v_2)))$ $(= int(conv_f(N_2 \cup \{v^*\})))$ as well, we have to choose a point $f'(v_2) \in D \cap int(conv_f(N_2 \cup \{v^*\}))$. This set is not empty, since $f(v^*) \in conv_f(N_H(v^*))$.

Therefore, by Lemma 5.2, if $f'(v_2)$ is chosen from $D \cap int(conv_f(N_2 \cup \{v^*\})) \cap B_f(v^*)$, then the resulted embedding $f'$ satisfies Definition 4.1(ii) for $H'$. Observe that $D \cap int(conv_f(N_2 \cup \{v^*\})) \cap B_f(v^*)$ is a non-empty open set, and hence such a choice (satisfying also that the points of $\{f'(v) \mid v \in V(H')\}$ are in general position) is possible. Therefore we can construct an SC-embedding of $H'$.

**Case (ii)**: $H = H' \triangle e$.

If $|N_{H'}(v_1)| \geq 4$ and $|N_{H'}(v_2)| \geq 4$ (which implies $H' \triangle e = H' - e$ and hence $V(H') = V(H)$), then $f$ is clearly an SC-embedding of $H'$. Hence we consider the case where $|N_{H'}(v_1)| = |N_{H'}(v_2)| = 3$ (the case when $|N_{H'}(v_1)| = 3$ and $|N_{H'}(v_2)| \geq 4$ can be treated similarly). Let $\{u_1, u_2\} = N_{H'-e}(v_1)$ and $\{u_3, u_4\} = N_{H'-e}(v_2)$. We may assume $u_1 \neq u_4$ without loss of generality.

We will define an SC-embedding $f'$ of $H'$ for which $f'(u) = f(u)$ for every $u \in V(H') - \{v_1, v_2\}$. Note that $u_i \in V(C)$ may hold for some $1 \leq i \leq 4$. To determine $f'(v_1)$ and $f'(v_2)$ we proceed as follows.

First we choose a point $x_1$ in the interior of the triangle of $f(u_1)$, $f(u_2)$, $f(u_4)$ (denoted by $T_1$) and a point $x_2$ in the interior of the triangle of $f(u_4), f(u_3), f(u_1)$ (denoted by $T_2$). Let $K_1$ ($K_2$) be the interior of the cone generated by the half lines $\hat{L}(f(u_1), x_1)$ and $\hat{L}(f(u_2, x_1)$ ($\hat{L}(f(u_4), x_2)$ and $\hat{L}(f(u_3), x_2)$, respectively). Let $S_4$ be an open disc with center $u_4$ in $K_1$ and let $S_1$ be an open disc with center $u_1$ in $K_2$. Observe that for every point $z$ in the interior of the triangle of $f(u_1), f(u_2), x_1$ (denoted by $R_1$) and for every $y \in S_4$ we have $z \in int(conv(\{f(u_1), f(u_2), y\}))$. Similarly, for every point $z'$ in the interior of the triangle of $f(u_4), f(u_3), x_2$ (denoted by $R_2$) and for every $y \in S_1$ we have $z' \in int(conv(\{f(u_4), f(u_3), y\}))$.

Let $U_1 = cone_f(u_1, u_2) \cap cone_f(u_2, u_1)$ and let $U_2 = cone_f(u_4, u_3) \cap cone_f(u_3, u_4)$. These are non-empty open sets. Clearly, $([f(u_1), f(u_2)] - \{f(u_1), f(u_2)\}) \subset U_1$ and $([f(u_4), f(u_3)] - \{f(u_4), f(u_3)\}) \subset U_2$ hold. For every point $w \in U_1$ ($w' \in U_2$) we have $u_1 \in int(conv(\{w\} \cup N_H(u_1) - \{u_2\}))$ and $u_2 \in int(conv(\{w\} \cup N_H(u_2) - \{u_1\}))$ ($u_4 \in int(conv(\{w'\} \cup N_H(u_4) - \{u_3\}))$ and $u_3 \in int(conv(\{w'\} \cup N_H(u_3) - \{u_4\}))$, respectively).

With these definitions we define the required positions as follows. Let $f'(v_1)$ be a point in $S_1 \cap U_1 \cap R_1$ and $f'(v_2)$ be a point in $S_4 \cap U_2 \cap R_2$, such that all the points in $\{f'(v) \mid v \in V(H')\}$ are in general position. It is easy to see that both $S_1 \cap U_1 \cap R_1$ and $S_4 \cap U_2 \cap R_2$ are non-empty open sets, and hence such a pair of points exists. Furthermore, it follows from the observations above, that $f'$ satisfies Definition 4.1(ii) for $H'$. Thus $f'$ is an SC-embedding of $H'$, as required.

The above arguments and Lemma 5.2 prove that given an SC-embedding $f$ of $H' \triangle e$ or $H'/e$, we can construct an SC-embedding $f'$ of $H'$ in $O(n \log n)$ time (note that a new point must be chosen so that no three points are on the same straight line). Since this procedure is executed $O(n)$ times to construct an SC-embedding of $G'$, the entire running time for finding an SC-embedding of the original graph $G$ is $O(n^2 \log n)$. ∎

We note that, given a set $P$ of $n$ points in general position and a non-empty open set $B \subseteq \mathbf{R}^2$, we can choose a new point $x \in B$ in $O(n \log n)$ time so that all points in $P \cup \{x\}$ are in general position. To do this we fix a new point $x_0 \in B$ temporarily, sort all the points $y \in P$ according to the gradient $g_{x_0}(y)$ of the line containing $y$ and $x_0$ in $O(n \log n)$ time (where we are done if no two points in $P$ have the same gradient), and move the position $x_0$ along a segment $[x_0, x'] \subseteq B$ whose gradient is different from any of $g_{x_0}(y)$, $y \in P$. There is an $\epsilon > 0$ such that for the position $x_1$ moved from $x_0$ by $\epsilon$ along $L$, the sorted order of $g_{x_1}(y)$ remains unchanged and no two points in $P$ have the same gradient (hence any point $x \in ([x_0, x_1] - x_0) \cap B$ is a desired solution). Such $\epsilon > 0$ can be determined in $O(n)$ time only by checking each pair of lines $L_i$ and $L_{i+1}$ with consecutive gradients $g_i < g_{i+1}$ in the sorted order.

## 6. Finding a 2-bisection in a 3-connected Graph

By combining the algorithmic proof of Theorem 5.3 and the ham-sandwich cut algorithm in two dimensions, we are now able to obtain a polynomial time algorithm that finds a (weak) 2-bisection in a 3-connected graph.

**Algorithm BISECTION** $(G, T_1, T_2)$
**Input**: A 3-connected graph $G$ and subsets $T_1, T_2 \subseteq V$ such that $|T_1|, |T_2|$ are even.
**Output**: A weak 2-bisection (resp., a 2-bisection, if $T_1 \cap T_2 = \emptyset$) $\{V_1, V_2\}$ of $(G, T_1, T_2)$.

1. Choose an arbitrary cycle $C$, and construct an SC-embedding $f$ of $G$ with boundary $C$.
2. Let $W = \{f(v) \mid v \in T_1\}$ and $B = \{f(v) \mid v \in T_2\}$. Compute a ham-sandwich cut $L$ with respect to $W$ and $B$. Let $L_1$ and $L_2$ denote the open half

planes defined by $L$. Let $V_i' = \{v \in V \mid f(v) \in L_i\}$, $W_i' = W \cap L_i$, $B_i' = B \cap L_i$, for $i = 1,2$. Let $M = \{f(v) \mid f(v) \in L\}$.

3. Find a bipartition $\{M_1, M_2\}$ of $M$ for which $|W_i' \cup (M_i \cap W)| \leq |W|/2+1$ and $|B_i' \cup (M_i \cap B)| \leq |B|/2+1$ (or $|W_i' \cup (M_i \cap W)| = |W|/2$ and $|B_i' \cup (M_i \cap B)| = |B|/2$, if $W \cap B = \emptyset$), for $i = 1,2$. Let $Q_i = \{v \in V \mid f(v) \in M_i\}$, $i = 1,2$.
4. Let $V_1 = V_1' \cup Q_1$ and let $V_2 = V_2' \cup Q_2$. Output the bipartition $\{V_1, V_2\}$ of $V$.

Since the points $\{f(u) : u \in V\}$ are in general position, we have $|M| \leq 2$ in Step 2. Therefore it is easy to see that the required bipartition $\{M_1, M_2\}$ exists in Step 3. By Lemma 4.2 (applied to the SC-embedding, the line $L$ and the sets $V_1' \cup Q_1$ and $V_2' \cup Q_2$), we obtain that $V_i$ induces a connected subgraph of $G$ for $i = 1,2$. Since $L$ is a ham-sandwich cut in Step 2, and by the choice of $M_1, M_2$, it follows that the output $\{V_1, V_2\}$ is a (weak) 2-bisection of $G$ with respect to $T_1$ and $T_2$.

An SC-embedding $f$ in Step 1 can be obtained in $O(n^2 \log n)$ time by Theorem 5.3. Steps 2 and 3 can be done in $O(n)$ time using the linear time ham-sandwich cut algorithm [2]. From the above discussion it follows that Conjecture 3.1 is true for $k = 2$:

**Theorem 6.1.** *Let $G = (V, E)$ be a 3-connected graph. Then there exists a weak 2-bisection in $G$ for every pair of specified sets $T_1, T_2$. If $T_1$ and $T_2$ are disjoint, then $G$ has a $k$-bisection. Such a (weak) bisection can be computed in $O(n^2 \log n)$ time.* ∎

It is conceivable that every 3-connected graph has a 2-bisection for every pair of (possibly intersecting) specified sets $T_1, T_2$. This would follow from our proof if the following possible strengthening of the planar ham-sandwich theorem held: if $P_1, P_2$ are point sets of odd cardinality in general position in $\mathbf{R}^2$ then there exists a ham-sandwich cut $L$ such that $|L \cap P_1| + |L \cap P_2| \leq 2$.

## 7. Remarks

In this section we make some remarks on Conjecture 3.1 for $k = 3$ and briefly discuss the directed graph versions of our problems. We also show that deciding whether a graph has an SC-embedding in the plane is NP-hard.

### 7.1. 3-bisections in 4-connected graphs

If a graph $G$ is isomorphic to the vertex-edge graph of some convex polytope in $\mathbf{R}^3$, then $G$ is called *polyhedral*. The following characterization of polyhedral graphs is well-known.

**Lemma 7.1.** [13] *A graph $G$ is a polyhedral graph if and only if $G$ is 3-connected and planar.* ∎

In order to prove Conjecture 3.1 for $k=3$ by a similar application of the (3-dimensional) ham-sandwich theorem (i.e. by using Theorem 2.1 and Lemma 4.2), it would be sufficient to prove that every 4-connected graph $G$ has an SC-embedding $f:V\to\mathbf{R}^3$. In such an embedding the boundary $G'$ should be a polyhedral subgraph of $G$ by Definition 4.1(i).

We conjecture that such an embedding exists (for every proper choice of the boundary).

**Conjecture 7.2.** For a 4-connected graph $G=(V,E)$ and any polyhedral subgraph $G'$ of $G$, there is an SC-embedding $f:V\to\mathbf{R}^3$ with boundary $G'$. ∎

As opposed to 3-connected graphs, it is not clear whether every 4-connected graph has a subgraph which can be chosen to be the boundary of an SC-embedding.

**Conjecture 7.3.** Every 4-connected graph $G=(V,E)$ has a 3-connected planar subgraph $G'=(V',E')$. ∎

The following weaker form of Conjecture 3.1 may also be interesting (and easier to prove). Let $G=(V,E)$ be $k$-connected with pairwise disjoint specified sets $T_1,\ldots,T_k$. Adding a new node $s$ to $G$ and connecting $s$ to each node $v\in V$ gives a $(k+1)$-connected graph. Therefore it would follow from Conjecture 3.1 that $G$ has a connected subgraph $H=(V',E')$ for which $|V'\cap T_i|=|T_i|/2$ for $1\le i\le k$. We conjecture that such a 'bisecting' connected subgraph exists in every $k$-connected graph (with respect to any choice of $k$ specified sets).

## 7.2. Bisections in directed graphs

The $k$-bisection problem can be formulated for directed graphs as well. In one possible version we may ask for a bipartition $\{V_1,V_2\}$ of the node set $V$ of a digraph $D=(V,E)$ such that $V_1$ as well as $V_2$ bisects $k$ given subsets $T_1,\ldots,T_k$ in such a way that $D[V_1]$ and $D[V_2]$ are both strongly connected. This condition seems to be too strong to impose, at least if one tries to extend Conjecture 3.1. For example, there exist 2-connected directed graphs without such 1-bisections.

There is a weaker definition, however, which allows a natural extension of Conjecture 3.1. Let us say that a partition $\{V_1,V_2\}$, bisecting each $T_i$, is a *directed $k$-bisection* in a directed graph $D=(V,E)$ if there exists a spanning arborescence in each of $D[V_1]$ and $D[V_2]$. With this definition, we can verify that every 2-connected directed graph has a directed 1-bisection. This follows easily from the existence of *directed $s$-$t$ numberings*, introduced by Cheriyan and Reif [1, Theorem 3.1]. A directed $s$-$t$ numbering of a 2-connected directed graph $D=(V,E)$ with two specified nodes $s$ and $t$ is a bijection $\pi$ between $V$ and $\{1,\dots,n\}$ such that $\pi(s)=1$, $\pi(t)=n$, and for each $v\in V-\{s,t\}$ there is a directed arc from $v$ to a node $w$ with $\pi(w)>\pi(v)$ and a directed arc from a node $u$ to $v$ with $\pi(v)>\pi(u)$.

Let $D=(V,E)$ be 2-connected and let $T\subseteq V$ be a specified set with $|T|=2p$. Take a directed $s$-$t$ numbering of $G$ and let $r$ be the maximum integer for which $|T\cap\{v\in V \mid \pi(v)\le r\}|\le p$. Clearly, each of the subgraphs induced by $V_1=\{v\mid \pi(v)\le r\}$ and $V_2=\{v\mid \pi(v)>r\}$ bisects $T$ and contains a spanning arborescence (rooted at $s$ and $t$, respectively). This settles the case $k=1$ of the following extension of Conjecture 3.1.

**Conjecture 7.4.** Let $D=(V,E)$ be a $(k+1)$-connected directed graph and $T_1,T_2,\dots,T_k$ be pairwise disjoint subsets of $V$, where $|T_j|$ is even for $j=1,\dots,k$. Then $G$ has a directed $k$-bisection.

### 7.3. NP-hardness of finding an SC-embedding of a graph

As shown by Theorem 5.3, the 3-connectivity is a sufficient condition for a graph $G$ to have an SC-embedding. In the rest of this section we show that the problem of testing whether an arbitrary graph $G$ admits an SC-embedding is NP-hard.

**Lemma 7.5.** *Let $G=(V,E)$ be a graph without isolated nodes and let $C$ be a cycle in $G$. If there is a subset $X\subseteq V-V(C)$ with $|N_G(X)|\le 2$, then $G$ has no SC-embedding with boundary $C$.*

**Proof.** For a contradiction suppose that $f$ is an SC-embedding of $G$ with boundary $C$. Let $|N_G(X)|=2$ for some $X\subseteq V-V(C)$ (the case $|N_G(X)|\le 1$ can be treated analogously) with $N_G(X)=\{v_1,v_2\}$. Let $L$ be the straight line that contains $f(v_1)$ and $f(v_2)$. Take a supporting hyperplane $L'$ (that is, a line) of $conv_f(X)$ parallel to $L$ and let $f(x)\in L'\cap conv_f(X)$ for some $x\in X$. Clearly, $f(x)\in int(conv_f(N_G(x)))$ cannot hold, a contradiction. ∎

If $G$ is not 2-connected, then for any cycle $C$ in $G$, there exists a subset $X\subseteq V-V(C)$ with $|N_G(X)|\le 1$, and hence a non-2-connected graph cannot have an SC-embedding by Lemma 7.5. In a 2-connected graph $G$, a subset

$X \subset V$ is called a *tight set* if $|N_G(X)| = 2$ and $V - X - N_G(X) \neq \emptyset$. In the 2-connected case, we have the next lemma.

**Lemma 7.6.** *Let $G = (V, E)$ be a 2-connected graph. Then $G$ has an SC-embedding if and only if there is a cycle $C$ in $G$ which satisfies $V(C) \cap X \neq \emptyset$ for every tight set $X$.*

**Proof.** If no cycle $C$ intersects all tight sets, then Lemma 7.5 says that $G$ cannot have an SC-embedding. Conversely, let cycle $C$ satisfy $V(C) \cap X \neq \emptyset$ for all tight sets $X$. Consider the graph $G' = (V \cup \{z\}, E')$ obtained from $G$ by adding a new node $z$ and arcs between $z$ and all nodes in $C$, where $E' = E \cup \{(z, v) \mid v \in C\}$. Note that $G'$ is 3-connected, and by Theorem 5.3 $G'$ has an SC-embedding $f$ with boundary $C$. Then we neglect $f(z)$ in $\{f(v) \mid v \in V \cup \{z\}\}$. By $N_{G'}(z) = C$, it is clear that the resulted embedding $\{f(v) \mid v \in V\}$ is still an SC-embedding of $G$. ∎

**Theorem 7.7.** *The problem of deciding whether $G = (V, E)$ has an SC-embedding is NP-hard.*

**Proof.** It is known that the problem of testing whether there exists a Hamiltonian cycle in a 3-regular and 3-connected graph $H = (W, F)$ is NP-complete [7]. Given an instance $H = (W, F)$ of the Hamiltonian cycle problem, we apply the following operation to each node $v \in W$. Let $N_G(v) = \{u_1, u_2, u_3\}$ for a node $v \in W$. We replace $v$ with three new nodes $v_1, v_2$ and $v_3$ and three arcs $(v_1, v_2)$, $(v_2, v_3)$ and $(v_3, v_1)$ which form a triangle, and let $N_G(v_1) = \{v_2, v_3, u_1\}, N_G(v_2) = \{v_3, v_1, u_2\}, N_G(v_3) = \{v_1, v_2, u_3\}$. Moreover we replace arc $(v_1, v_2)$ with two arcs $(v_1, w_v)$ and $(v_2, w_v)$ introducing a new node $w_v$. Let $H'$ be the resulted graph obtained by applying this operation to all nodes $v \in W$. Note that for each $v \in W$, $\{w_v\}$ is a tight set in $H'$, and there are no other tight sets. Then it is easy to see that $H'$ has a cycle $C$ such that $V(C) \cap X \neq \emptyset$ for every tight set $X$ if and only if $H$ has a Hamiltonian cycle. It is also obvious that this transformation can be done in polynomial time. ∎

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## References


[1] J. Cheriyan and J. H. Reif: Directed $s$–$t$ numberings, rubber bands, and testing digraph $k$-vertex connectivity, *Combinatorica,* 14(4) (1994), 435–451.

[2] L. Chi-Yuan, J. Matoušek and W. Steiger: Algorithms for ham-sandwich cuts, *Discrete Comput. Geom.,* **11** (1994), 433–452.

[3] J. Chleíková: Approximating the maximally balanced connected partition problem in graphs, *Information Processing Letters,* **60** (1996), 225–230.

[4] M. E. Dyer and A. M. Frieze: On the complexity of partitioning graphs into connected subgraphs, *Discrete Applied Mathematics,* **10** (1985), 139–153.

[5] H. Edelsbrunner: *Algorithms in Combinatorial Geometry,* Springer-Verlag, Berlin, 1987.

[6] H. Edelsbrunner and R. Waupotitsch: Computing a ham-sandwich cut in two dimensions, *J. Symbolic Computation,* **2** (1986), 171–178.

[7] M. R. Garey, D. S. Johnson and R. E. Tarjan: The planar Hamiltonian circuit problem is NP-complete, *SIAM J. Comput.,* **5** (1976), 704–714.

[8] E. Győri: On division of connected subgraphs, *Combinatorics, (Proc. Fifth Hungarian Combinatorial Coll, 1976, Keszthely),* Bolyai-North-Holland, 1978, 485–494.

[9] J. Hopcroft and R.E. Tarjan: Dividing a graph into triconnected components, *SIAM J. Comput.,* **2** (1973), 135–158.

[10] N. Linial, L. Lovász and A. Wigderson: Rubber bands, convex embeddings and graph connectivity, *Combinatorica,* **8** (1988), 91–102.

[11] L. Lovász: A homology theory for spanning trees of a graph, *Acta Math. Acad. Sci. Hungar,* **30** (1977), 241–251.

[12] H. Nagamochi and T. Ibaraki: A linear-time algorithm for finding a sparse $k$-connected spanning subgraph of a $k$-connected graph, *Algorithmica,* **7** (1992), 583–596.

[13] E. Steinitz: *Polyeder und Raumeinteilungen,* Encyklopädie der mathematischen Wissenschaften, Band III, Teil 1, 2. Hälfte, IIIAB12, 1916, 1–139.

[14] H. Suzuki, N. Takahashi and T. Nishizeki: A linear algorithm for bipartition of biconnected graphs, *Information Processing Letters,* **33** (1990), 227–232.

[15] W. T. Tutte: *Connectivity in Graphs,* University of Toronto Press, 1966.

[16] K. Wada and K. Kawaguchi: Efficient algorithms for tripartitioning triconnected graphs and 3-edge-connected graphs, *Lecture Notes in Comput. Sci.,* 790, Springer, Graph-theoretic concepts in computer science, 1994, 132–143.

[17] K. Wada, K. Kawaguchi and N. Yokoyama: On a generalized $k$-partition problem for graphs, *Proc. 6th Karuizawa Workshop on Circuit and Systems,* 1993, 243–248.

[18] G. M. Ziegler: *Lectures on Polytopes,* Springer-Verlag, Graduate Texts in Mathematics 152, 1995.

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